The Relationship Between Shoe Size and the Amount of Sand that Enters the Shoe

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ABSTRACT

This study examines how shoe sizes affect the amount of sand that enters the shoe. By building a simple mathematical model that compares ankle size and ankle-collar size, we show that sand inflow can be approximated as proportional to the difference between the area of the shoe opening and the cross-sectional area of the ankle. For non-custom-made shoes produced in fixed sizes, the model suggests that size intervals may affect sand intake because larger size intervals can create larger ankle gaps. A small pilot observation using five shoe cases is included to connect the theoretical model to non-hypothetical data. The results do not provide a directly prescriptive industrial recommendation, but they suggest a conceptual way to think about shoe-size intervals, ankle-collar geometry, and wearer comfort.

KEYWORDS

shoe size, sand inflow, mathematical modeling, exponential scaling

SECTION 1. INTRODUCTION

Oftentimes, while walking along a beautiful nature trail, we find ourselves slightly annoyed by sand getting into our shoes. In this study, we examine how shoe size affects the amount of sand that enters the shoe.

Figure 1. When running in sandy environments like the beach, our shoes keep filling with sand.
(photo credit [1])

Because the width of the foot is larger than the ankle, the opening of a shoe must be wider than the ankle in order for the foot to enter. As a result, the shoe opening can function like a funnel. Even when a shoe fits reasonably well, sand can still enter because the ankle does not perfectly fill the opening.

Unless shoes are custom-made, shoes almost always come in fixed sizes. It becomes difficult for companies to manage inventory if sizes are too varied, but customers have fewer options for a good fit if the size categories are too broad. In the final section of this study, we discuss how the amount of sand entering shoes can be used as a conceptual basis for thinking about shoe-size intervals.

This study is organized as follows. In Section 2, we propose a mathematical model for the shape of a shoe and a human foot and include a small pilot observation. In Section 3, we use this model to calculate the amount of sand that enters the shoe. In Section 4, we apply the model to shoe-size intervals and derive a condition on how size differences might scale with shoe size. Finally, in Section 5, we summarize the conclusions and discuss limitations.

SECTION 2: MATHEMATICAL MODEL OF SHOES AND FEET

In this section, we build a mathematical model of shoes and feet. Consider two pairs of shoes where the only difference between them is size. In mathematical terms, the two shoes are geometrically similar. The size of the shoe is proportional to the length of the foot that fits inside. Thus, the similarity ratio of the shoes is equal to the ratio of their sizes.

The shoe size becomes the single parameter that represents the shape of the shoe. We denote this parameter by s. The length, width, and height of the shoe are all proportional to s. Since areas scale with the square of the similarity ratio, the area of the shoe opening and the cross-section around the ankle are proportional to s^2. Following the same logic, volumes scale with the cube of the similarity ratio, and therefore, the volume of the shoe is proportional to s^3.

Wearing the correct shoe size indicates that both the foot and ankle fit the shoe properly. However, if the ankle collar were the same size as the actual ankle, it would be very difficult to insert the foot. Therefore, the ankle collar should be larger than the ankle. This results in the shoe opening taking the shape of a funnel. This situation can be illustrated as in the following figure.

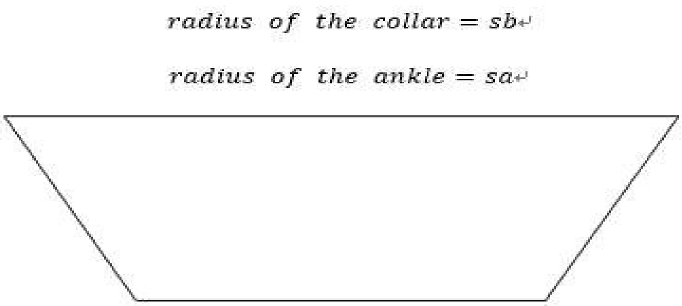

Figure 2. The geometric relation between the ankle collar and the ankle size.

The ratio between the ankle radius and the ankle-collar radius is determined by constants. Let the model ankle radius be sa, and the model ankle-collar radius be sb for some positive constants a and b. Since the ankle collar is larger than the ankle itself, we assume

$$sa < sb$$

which leads to

$$a < b$$

Now, the modeling of the shoe is complete. Next, we model the human foot to link it with the shoe. Let a person's ankle radius be r. Then the shoe size s that the person can wear must satisfy

$$sa \geq r$$

This is because the ankle-related dimension of the shoe must be greater than or equal to the actual ankle radius for the person to be able to put the shoe on. Thus, for a person with an ankle radius r, the minimum shoe size that can be worn is defined as

$$s_0 = r / a$$

Naturally, this value depends on the ankle radius r and the design parameter a of the shoe. The model parameter sb is used for the theoretical ankle-collar radius. In the pilot data below, R is used for the measured ankle-collar radius so that the observational measurements are clearly distinguished from the model expression.

Preliminary Observational Data

To support the mathematical model with non-hypothetical observations, I conducted a small pilot measurement using several pairs of shoes and ankle measurements. For each case, I recorded shoe size, approximate ankle radius, approximate ankle-collar radius, and the resulting gap area between the ankle and the shoe opening.

The purpose of this pilot observation was not to provide a complete experimental validation, but to check whether the model's central assumption-that a larger gap between the ankle and ankle collar allows more sand inflow-is reasonable in real examples.

For the observational table, r is the measured ankle radius and R is the measured ankle-collar radius. The estimated gap area is calculated as:

$$\pi(R^2 - r^2)$$

Case	Shoe Size	Ankle Radius r (cm)	Ankle-Collar Radius R (cm)	Estimated Gap Area $\pi(R^2 - r^2)$ (cm^2)	Observed Sand Inflow
1	235	3.1	3.3	4.02	Low
2	245	3.2	3.4	4.15	Low-Mid
3	255	3.2	3.6	8.55	Mid
4	270	3.4	3.7	6.69	Mid
5	285	3.5	4.0	11.78	High

Although the sample size is limited, the observations connect the theoretical model to measurable physical quantities. The cases with relatively larger estimated gap areas generally show higher observed sand inflow. This provides preliminary support for the model's basic assumption. However, because the observation is small in scale and qualitative, additional variables such as walking speed, shoe material, sock thickness, sand grain size, and ground condition should be included in future versions of the model.

SECTION 3: CALCULATION OF SAND INFLOW

In the previous section, we proposed a mathematical model for the sizes of shoes and feet and included a small pilot observation. In this section, we use the model to discuss the amount of sand entering the shoe.

The amount of sand that enters is assumed to be proportional to the difference between the area of the ankle-collar opening and the cross-sectional area of the ankle. This is because sand can pass through this gap and eventually accumulate inside the shoe. We model the ankle collar as a circle of radius sb. Its cross-sectional area is

$$\pi(sb)^2 = \pi s^2 b^2$$

The cross-sectional area of the ankle itself can also be modeled as a circle with radius r, so its area is

$$\pi r^2$$

Therefore, the difference in areas is

$$\pi(s^2 b^2 - r^2)$$

If we let A denote the amount of sand entering the shoe per unit time, this can be expressed as

$$A = \pi k(s^2 b^2 - r^2)$$

k is a constant that depends on environmental conditions, including ground surface, sand density, and sand particle size. On sandy playgrounds or beaches, k will be large; on marble floors or asphalt roads, it will be much smaller. Prior research on sand particle size also suggests that sand properties vary across environments, which supports treating k as an environmental parameter [4].

In this study, however, we do not vary k for each type of ground. Instead, we define k as an averaged value representing typical sandy surfaces. This choice does not create a perfect model, but it is sufficient to describe the general relationship between ankle-collar gap area and sand inflow.

Using the minimum shoe size $s_0 = r/a$, we can rewrite $r = as_0$. Therefore, when a person wears a shoe of size s, the amount of sand entering the shoe can be expressed as

$$A = \pi k(s^2 b^2 - s_0^2 a^2)$$

Since $s \geq s_0$, the minimum value of A occurs when $s = s_0$. Thus,

$$A_{min} = \pi k(s_0^2 b^2 - s_0^2 a^2) = \pi k s_0^2 (b^2 - a^2)$$

From the assumption about shoe shapes, we know that b > a. This makes the minimum value of A positive. In other words, sand may still enter shoes even when the shoe size is appropriately matched to the wearer, because the ankle collar must remain wider than the ankle itself. This theoretical result is consistent with the preliminary observational data presented in Section 2, where larger measured gap areas generally corresponded to higher observed sand inflow.

SECTION 4: APPLICATION TO SHOE SIZE INTERVALS

In the previous section, we showed that shoes may allow sand to enter through the gap between the ankle and the ankle collar. In mass-produced shoes, the situation becomes more complicated because shoes are produced in fixed sizes rather than individually customized dimensions. International footwear sizing systems, such as the Mondopoint system, also show that shoe sizing is commonly organized around standardized measurements [3].

Let the minimum shoe size that a person can wear be s_0. If a shoe company produces sizes with an interval of d, the worst case for a person who should wear size s0 would be to wear size

$$s_0 + d$$

This happens when size s_0 is just slightly too small, requiring the wearer to size up to the next available option. By substituting $s = s_0 + d$ into the formula for sand inflow, we obtain

$$A = \pi k((s_0 + d)^2 b^2 - s^2 a^2)$$

Since $s_0 + d > s_0$ and $b > a$, A is positive. It is also an increasing function of d. This means that larger shoe-size intervals may create larger gaps around the ankle and therefore allow more sand to enter.

We now make one additional assumption. If the amount of sand entering the shoe becomes too large, the wearer's discomfort increases. For the same amount of sand, a person with smaller feet may feel more discomfort because the sand is distributed across a smaller foot area. Therefore, we assume that the maximum tolerable amount of sand is proportional to the area of the foot, which can be expressed as

$$cs_0^2$$

Here, c is a proportionality constant representing the wearer's tolerance for sand discomfort. Thus, the size interval d must satisfy

$$\pi k((s_0 + d)^2 b^2 - s_0^2 a^2) \leq cs^2$$

This can be simplified to

$$\pi k(s_0 + d)^2 b^2 \leq s^2 (c + \pi k a^2)$$

Taking the square root gives

$$s_0 + d \leq (s / b) \sqrt{(c/\pi k + a^2)}$$

Therefore,

$$0 < d \leq s_0((1/b)\sqrt{(c/\pi k + a^2)} - 1)$$

This result shows that the maximum possible value of d is proportional to s_0. In other words, under the assumptions of this model, shoe-size intervals should not necessarily increase by the same fixed amount. Larger shoe sizes could allow larger intervals, while smaller shoe sizes may require smaller intervals in order to control sand inflow.

This does not mean that manufacturers should immediately adopt this model. Rather, the model provides a conceptual way to think about how physical factors, such as ankle-collar gap area and sand inflow, could be considered when designing shoe-size intervals.

SECTION 5: CONCLUSION

In this paper, we developed a simple mathematical model relating shoe size, ankle size, and the amount of sand that enters a shoe. The model assumes that sand inflow is proportional to the gap area between the ankle and the ankle collar.

The theoretical model suggests that, if sand inflow is considered as a design factor, shoe-size intervals may be better represented by proportional scaling rather than equal linear increments. In other words, larger shoe sizes may allow larger intervals, while smaller shoe sizes may require smaller intervals to maintain similar comfort levels.

The preliminary observational data provided limited non-hypothetical support for this idea. In the small pilot measurement, cases with larger estimated gap areas generally showed higher observed sand inflow. However, the sample size was small, and the observation was qualitative rather than fully experimental.

The conclusions in this work are theoretical and rely on simplifying assumptions. Therefore, the results should not be taken as directly prescriptive for industry. Instead, they provide a starting point for thinking about how physical factors, such as ankle-collar gap area, sand inflow, and shoe-size intervals, could be integrated into future studies of footwear design.

Future research should include a larger sample size, controlled sand conditions, repeated walking trials, and more precise measurements of ankle-collar geometry. Additional factors such as walking speed, shoe material, sock thickness, and sand particle size should also be considered.

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